Correlations in the scattering cross section of regular systems

T Gorin†

Max Planck Institut für Physik Komplexer Systeme, D-01187 Dresden, Germany

Abstract. Recently, several authors recognized and discussed the fact that regular closed quantum systems, i.e., systems which do not show correlations in their level spectrum, are nevertheless likely to exhibit correlations in the scattering cross section if they are coupled with sufficient strength to decay channels.

In this paper a new averaging method is developed, especially designed to deal with S-matrices constructed on the basis of Hamiltonians with uncorrelated eigenvalues. The method is applied to the correlation function of the S-matrix in the one-channel case. It leads to a two-dimensional integral representation of the correlation function which is accessible to an asymptotic series expansion.

Depending on the statistical properties of the coupling matrix elements a 'correlation hole' appears in the real part of the correlation function. It becomes fully developed in the strong coupling limit, but in some cases already appears before the coupling strength has reached the point of maximal transmission (critical point).

1. Introduction

Many quantum experiments can be viewed as the opening of an originally closed system due to a measurement process. The formerly discrete spectrum then becomes a continuous one, and the eigenstates turn into resonances. A simple example would be a microwave billiard [1–3]. Due to the equivalence with a two-dimensional quantum billiard, such a system may be used to study fundamental questions concerning transport phenomena in mesoscopic systems [4,5].

In order to analyse the eigenstates of a microwave billiard experimentally, an antenna (usually a small lead) is placed inside. Via this antenna a microwave is injected, and its reflection coefficient is measured and recorded as a function of the energy (frequency). It gives the scattering cross section [6, 7] for this one-channel scattering system (the number of channels is given by the number of antennas).

Depending on the shape of the billiard the classical motion can be chaotic or integrable [8]; usually integrable and chaotic motion coexist in distinct areas of the phase space. In general the type of classical motion is reflected in the correlation properties of the level spectrum of the corresponding quantum system. Integrable systems typically show no correlations in the spectrum (Poissonian spectrum), whereas chaotic systems typically show the statistical properties of the so-called Gaussian ensembles [9].

Naturally, the question arises, of how the statistical properties of the level spectrum are transferred to the scattering cross section. The transfer is governed by the coupling mechanism

† Present address: Centro Internacional de Ciencias, AP 6-101, 62131 Cuernavaca, Mexico. E-mail address: gorin@ce.ifisicam.unam.mx

between the discrete eigenstates and the decay channels. As long as the coupling is weak, first-order perturbation theory may be used. In this approximation the eigenvalues get an additional width and turn into resonances without changing their positions. All the correlations occurring in the scattering cross section, originate from the level spectrum. For the non-perturbative regime this is no longer true and more powerful methods for the calculation of correlations are needed and have been developed [7, 10–13]. However, the analytical results obtained there, mostly apply to Gaussian ensembles only.

This paper considers Poissonian ensembles, i.e., ensembles of systems with uncorrelated levels. This means that there are no correlations to be transferred to the scattering cross section. Nethertheless, numerical investigations have shown that correlations may show up [6, 13–15]. In such a case, the strong coupling not only transfers, but produces, correlations. The averaging method for Poissonian ensembles developed in this paper allows one to trace this process analytically. It leads to a two-dimensional integral representation of the correlation function of the S-matrix. The corresponding expression is then expanded in an asymptotic series. This gives a clue for the detection of correlations possibly present in the scattering cross section.

This paper is organised as follows. In section 2 the scattering system and the Poissonian scattering ensemble are defined. In section 3 the average S-matrix and related quantities are introduced. The nomenclature follows the optical model developed to describe compound-nuclear reactions [16]. Section 4 gives an outline of the new ensemble-averaging method. In section 5 the method is used to calculate the correlation function of the S-matrix for a Poissonian ensemble. The asymptotic series expansion of the correlation function is derived in section 6. In section 7, a particular case is investigated, where the coupling to the continuum produces very strong correlations. Section 8 consists of a summary.

2. Scattering ensemble

The starting point for constructing the scattering ensemble is the definition of the S-matrix in terms of finite-dimensional matrices H_0 and V:

$$S(E) = 1 - iV^\dagger \frac{1}{E - H} V \tag{1}$$

$$H = H_0 - \frac{i}{2} V V^\dagger. \tag{2}$$

H_0 is a real $N \times N$ matrix, describing the closed quantum system. V is a real $N \times K$ matrix, coupling each of the eigenstates of H_0 to each of the K decay channels. The rows of V are called channel vectors. They span a K-dimensional subspace through which the wavefunction of the system decays. H is called the effective Hamiltonian. Its complex eigenvalues are the poles of the S-matrix and give the positions and widths of the resonances [16].

One way to arrive at such a representation of the S-matrix is the Feshbach-projection formalism [17, 18], developed in order to describe compound-nuclear reactions. A completely different class of systems for which S-matrices of form (1) can be derived are microwave billiards with leads [6].

The S-matrix can also be expressed in the so-called K-matrix representation [16]:

$$S(E) = \frac{1 - iK(E)}{1 + iK(E)} \tag{3}$$

$$K(E) = \frac{1}{2} V^\dagger \frac{1}{E - H_0} V. \tag{4}$$

As the K-matrix is hermitian, (3) explicitly shows the unitarity of the S-matrix.

2.1. Definition of the random matrices H_0 and V

The S-matrix (1) is constructed with only a finite number of real parameters, the matrix elements of H_0 and V. The construction of the scattering ensemble is therefore accomplished by defining H_0 and V as random matrices. Dealing with Poissonian systems it is most convenient to work in the eigenbasis of H_0 so that the number of random variables is further reduced to the set of uncorrelated eigenvalues of H_0. Their distribution function $\rho(\varepsilon)$ is normalized

$$\int \rho(\varepsilon)\,\mathrm{d}\varepsilon = 1 \tag{5}$$

and equal to the level density, divided by N. This means that the local mean level distance $d(\varepsilon)$, defined by the inverse level density, is of the order of N^{-1}:

$$d(\varepsilon) = \frac{1}{N\rho(\varepsilon)}. \tag{6}$$

Concerning the coupling matrix V, it is desirable to allow different coupling strengths for the various channels. This is done by introducing additional parameters $\eta_1, \ldots, \eta_K$ and a new matrix v such that

$$V_{ja} = \sqrt{\eta_a}\, v_{ja}. \tag{7}$$

The matrix elements v_{ja} are assumed to be independent random variables with the distribution function $p(v)$, a symmetric function with the following normalization properties:

$$\int p(v)\,\mathrm{d}v = 1 \qquad \int v^2 p(v)\mathrm{d}v = N^{-1}. \tag{8}$$

3. Average S-matrix and related quantities

Here, the much simpler spectral averages (averaging over many resonances in a small energy interval) are used. The mean S-matrix gives:

$$\langle S(E)\rangle = \frac{1 - \mathrm{i}\langle K(E^+)\rangle}{1 + \mathrm{i}\langle K(E^+)\rangle} \qquad E^+ = E + \mathrm{i}\delta \tag{9}$$

with an infinitesimal δ. In order to prove (9), one solves (3) for the K-matrix. The resulting expression is an analytic function of the S-matrix. As for any such function $f(S)$, $\langle f(S)\rangle = f(\langle S\rangle)$ holds [10], this is also true for the K-matrix.

In order to also apply (9) to ensemble averages, one has to assume that ensemble and spectral averages are equivalent ('ergodicity hypothesis' [19]). For the formulae presented in this section, 'ergodicity' can be proven explicitly for many different scattering ensembles. For Poissonian ensembles, this is done in [20] applying the method presented in section 4.

Average K-matrix.   In the eigenbasis of H_0 each element of the K-matrix reads:

$$K_{ab}(E^+) = \frac{1}{2} \sum_{j=1}^{N} \frac{V_{ja} V_{jb}}{E^+ - \varepsilon_j}. \tag{10}$$

As the different terms in the sum are all independent, the averaging can be performed easily. Cauchy's theorem [21] leads to

$$\langle K_{ab}(E^+)\rangle = \delta_{ab} \frac{\eta_a}{2} \left[\mathcal{P} \int \mathrm{d}\varepsilon \, \frac{\rho(\varepsilon)}{E - \varepsilon} - \mathrm{i}\pi\rho(E) \right]. \tag{11}$$

The non-diagonal matrix elements of the average K-matrix vanish. Assuming a symmetrical level density the mean K-matrix in the centre of the spectrum becomes purely imaginary, because there, the principal-value integral vanishes:

$$\langle K_{aa}(0^+)\rangle = -\mathrm{i}\frac{\pi}{2}\eta_a\rho(0) = -\mathrm{i}\kappa_a. \tag{12}$$

The second equality defines the coupling parameters κ_a, the appropriate dimensionless parameters needed to measure the coupling strength of the system to the various channels.

Inserting (12) into (9) the mean S-matrix in the centre of the spectrum simplifies to:

$$\langle S_{aa}(0)\rangle = \frac{1-\kappa_a}{1+\kappa_a}. \tag{13}$$

Transmission coefficients.   For a nuclear reaction the transmission coefficient for channel a describes the probability that the projectile from channel a gets trapped by the target nucleus and suffers a slow quasi-statistical scattering under formation of a compound nucleus [16]. In the simple picture used here, the only alternative process is shape elastic scattering, are described by the average S-matrix. Therefore:

$$\mathcal{T}_a = 1 - \langle S_{aa}(0)\rangle^2 = \frac{4\kappa_a}{(1+\kappa_a)^2}. \tag{14}$$

This formula shows that for small values of κ_a, the transmission $\mathcal{T}_a$ is proportional to the coupling strength κ_a, whereas for large values it is proportional to its inverse κ_a^{-1}. The point at which the transmission becomes maximal $\kappa_a = 1$ is referred to as the 'critical point' [20]. It is equal or at least closely related to the 'critical point' defined in connection with the 'trapping effect' [15, 22–25]. For each channel a one broad resonance separates from all others in that moment, when its coupling parameter κ_a exceeds the critical point. The broad resonance separates from all others collecting more and more of the available width, and finally it dominates the cross section globally. In doing so it gets absorbed into the shape elastic part of the cross section and disappears from its fluctuating part, for which the natural scale is the mean level distance.

In all that follows only the one-channel case will be considered. Then the S-matrix and the K-matrix are simple meromorphic functions of the energy and there is only one parameter κ for the coupling strength and only one transmission coefficient $\mathcal{T}$.

4. Averaging method for products of S-matrices

Here the new method is presented, which allows one to average products of S-matrices over a Poissonian ensemble as defined in section 2. The method is based on representing the S-matrix as linearly depending on an exponential of the K-matrix, which should be expressed in the eigenbasis of H_0 (10). As the K-matrix is a sum of statistically independent terms, the integration over the ensemble factorizes accordingly:

$$S(E) = \int_0^\infty \mathrm{d}r\, \mathrm{e}^{-r^2/2}(r + \partial_r)\mathrm{e}^{-\mathrm{i}r^2 K/2} = L_r \mathrm{e}^{-\mathrm{i}r^2 K/2}. \tag{15}$$

L_r is a linear functional. The existence of the ensemble average $\langle \mathrm{e}^{-\mathrm{i}r^2 K/2}\rangle$ is presupposed, one can do the averaging first and then apply L_r. $\mathrm{e}^{-\mathrm{i}r^2 K/2}$ factorizes in a product of N exponentials with independent integration variables ε_j and v_j, and so does the ensemble average:

$$\langle S(E)\rangle = L_r \langle Z\rangle_{\varepsilon,v}^N \qquad Z = \exp\left[-\frac{\mathrm{i}\eta v^2}{4}\frac{r^2}{E-\varepsilon}\right]. \tag{16}$$

One is left with an average over only two random variables: ε and v, which represent a diagonal element of H_0 or a normalized coupling matrix element, respectively. Their distribution functions are denoted by $\rho(\varepsilon)$ and $p(v)$ (cf section 2).

Analogously the average of a product of S-matrices (one being complex conjugated) gives:

$$\langle S(E)S^*(E')\rangle = L_r L_s \langle Z^-\rangle^N \qquad Z^- = \exp\left[-\frac{i\eta v^2}{4}\left(\frac{r^2}{E-\varepsilon} - \frac{s^2}{E'-\varepsilon}\right)\right]. \tag{17}$$

This is the central equation for the calculation of the correlation function. The formerly $2N$-dimensional integral is reduced to a two-dimensional one, which is then taken to the power of N.

In a similar way, any finite number of S-matrices can be considered. One may always profit from the reduction of integration variables, although the remaining integrals become increasingly complicated. However, if none of the S-matrices in the product is complex conjugated, then the average over the product is equal to the product of the individually averaged S-matrices and therefore need not be discussed here. The reason lies in the analytic properties of the S-matrix (cf the discussion below (9)).

Being interested in the case of a large number of resonances N, we consider the ensemble average as $N \to \infty$. In section 5, it will be shown that the expansion of $\langle Z^-\rangle_\varepsilon$ in powers of N^{-1} leads to

$$\langle Z^-\rangle_\varepsilon = 1 - c_0^-(v)N^{-1} + O(N^{-2}). \tag{18}$$

The relation $f(x) = O(x^k)$ is always used in combination with a limit $x \to x_0$. Then, it signifies that there exists an interval U_0 around x_0, of finite length, and a constant $A < \infty$ such that $\forall x \in U_0 : |f(x)| < Ax^k$.

Averaging (18) over the remaining random variable v (representing a coupling matrix element), it follows that in the limit $N \to \infty$, $\langle Z^-\rangle^N$ is equal to $\exp(-\langle c_0\rangle_v)$. Summarizing, one gets the following equation for the average of a product of two S-matrices:

$$\langle S(E)S^*(E')\rangle = L_r L_s \exp(-\langle c_0^-\rangle_v) \qquad c_0^-(v) = \lim_{N\to\infty} N(1 - \langle Z^-\rangle_\varepsilon). \tag{19}$$

So, the major task consists in calculating $c_0^-(v)$.

5. S-matrix correlations

In this section the correlation function of the S-matrix will be calculated. It is defined as follows:

$$C^-[S](\omega) = \langle S(d_0\omega/2)S^*(-d_0\omega/2)\rangle - |\langle S(0)\rangle|^2. \tag{20}$$

Usually [11, 20, 30] the correlation function is defined to be the complex conjugate of (20). The advantage of the definition here, is a possitive imaginary part of $C^-[S](\omega)$, which allows one to plot real and imaginary parts in a single diagram (cf figures 2–4). $C^-[S](\omega)$ measures the correlations between the values of two S-matrix elements taken at energies at a distance of ωd_0 in the centre of the spectrum. $d_0 = d(0)$, where $d(\varepsilon)$ is the mean level distance defined in (6). $C^-[S](\omega)$ is constructed in such a way, that its argument ω is of the order of one. Being interested in the limit $N \to \infty$, one can neglect the shifts $\pm d_0\omega/2$ in the arguments of the mean S-matrices and replace $\langle S(0)\rangle$ by (12). As described in section 4 the representation (17) is used for the average of the product of S-matrices. Altogether this leads to:

$$C^-[S](\omega) = L_r L_s \lim_{N\to\infty} \langle Z^-\rangle_{\varepsilon,v}^N - \left(\frac{1-\kappa}{1+\kappa}\right)^2 \tag{21}$$

$$Z^- = \exp\left[\frac{i\eta v^2}{4}\left(\frac{r^2}{d_0\omega/2-\varepsilon} - \frac{s^2}{-d_0\omega/2-\varepsilon}\right)\right]. \tag{22}$$
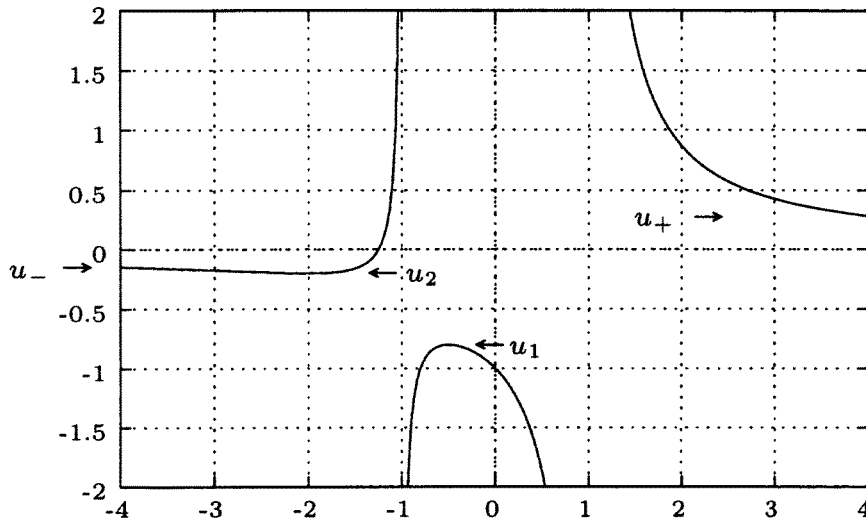
Figure 1. $u(x)$, equation (27) with $\tau = 0.8$.

In the following it is assumed that the level density is constant. This is justified because the level density varies on a scale which is N times larger than the level distance, which is the scale of $C^-[S](\omega)$.

$$\rho(\varepsilon) = \frac{1}{2a} \begin{cases} 1 & |\varepsilon| < a \\ 0 & \text{elsewhere} \end{cases} \qquad d = \frac{2a}{N}. \tag{23}$$

It is convenient to start with the integration over the level density. After a change of the integration variable $\varepsilon = d\omega x/2$ one has

$$\langle Z^- \rangle_\varepsilon = \frac{\omega}{2N} \int_{-N/\omega}^{N/\omega} dx \, \exp\left(ip\frac{\tau x + 1}{x^2 - 1}\right) \tag{24}$$

where the following notations were used:

$$p = 2\alpha(r^2 + s^2) \qquad \alpha = \frac{\tilde{\kappa}}{2\pi\omega} \qquad \tilde{\kappa} = \frac{\pi\eta v^2}{2d} \qquad \langle \tilde{\kappa} \rangle_v = \kappa \tag{25}$$

$$\tau = \frac{r^2 - s^2}{r^2 + s^2} \qquad \sigma = \frac{2rs}{r^2 + s^2} \qquad \tau^2 + \sigma^2 = 1. \tag{26}$$

Note, that the integral in (24) is invariant under changing the sign of τ. Therefore, only positive τ are considered, and at the end of the calculations, τ is replaced by $|\tau|$. Now, choose the x-dependent part of the exponent as the new integration variable:

$$u(x) = \frac{\tau x + 1}{x^2 - 1}. \tag{27}$$

This function is plotted for the value $\tau = 0.8$ in figure 1. It has two poles at ± 1 and two maxima where it reaches the values u_1 and u_2. Therefore, the range of the integral has to be split into five parts, in each of which $u(x)$ is strictly monotonous:

$$X^\pm(u) = \frac{1}{2u}\left(\tau \pm \sqrt{\tau^2 + 4u(u+1)}\right). \tag{28}$$

In figure 1 the finite limits of the image-intervals are marked with arrows. They have the following values:

$$u_{1/2} = -\tfrac{1}{2}\left(1 \pm \sqrt{1-\tau^2}\right) \qquad u_{+/-} = \omega\frac{1 \pm N\tau}{N^2 - \omega^2} = \pm\frac{\omega\tau}{N} + \mathrm{O}(N^{-2}). \tag{29}$$

The correct sign in each integral can be found by checking the asymptotic behaviour of $X^{\pm}(u)$ at the limits. Note, that the integral twice runs over the intervals $(-\infty, u_1)$ and (u_+, ∞). Finally we get

$$\langle Z^-\rangle_\varepsilon = \frac{\omega}{2N}\left\{\left[\int_{-\infty}^{u_1} - \int_{u_2}^{u_-} - \int_{u_+}^{\infty}\right] \mathrm{d}u\, g(u) + \int_{u_-}^{u_+} \mathrm{d}u\, f(u)\right\}\mathrm{e}^{\mathrm{i}pu} \tag{30}$$

$$g(u) = -\frac{\tau^2 + 2u}{u^2\sqrt{\tau^2 + 4u(u+1)}} \tag{31}$$

$$f(u) = -\frac{\tau}{2u^2} + \frac{1}{2u^2}\sqrt{\tau^2 + 4u(u+1)} - \frac{2 + 1/u}{\sqrt{\tau^2 + 4u(u+1)}}. \tag{32}$$

Next, according to the outline in section 4, $\langle Z^-\rangle_\varepsilon$ should be cast into the form $\langle Z^-\rangle_\varepsilon = 1 - c_0^- N^{-1} + \mathrm{O}(N^{-2})$. Due to the pre-factor, the integrals within the curly brackets should amount to a term $2N/\omega$, a constant term, and further negligible terms of the order of N^{-1}.

Consider the integral containing the function $f(u)$ in its integrand. Due to its boundaries of $\mathrm{O}(N^{-1})$ and due to f being of $\mathrm{O}(1)$ within these boundaries, it can be entirely neglected. $g(u)$ contained in the integrand of the remaining integrals has a pole at $u = 0$. The pole makes the value of the integral diverge linearly with N. In order to separate the divergent term, we introduce the function

$$h(u) = -\tau\,\mathrm{sgn}(u - u_1)u^{-2}. \tag{33}$$

Then, while $h(u)\mathrm{e}^{\mathrm{i}pu}$ can be integrated analytically, $g'(u) = g(u) - h(u)$ has no more pole at $u = 0$, so that the integration intervals (u_2, u_-) and (u_+, ∞) can be combined into one. The sgn-function in (33) is used in order to have $\lim_{\tau\to1} g(u) = h(u)$. It leads to simpler expressions in the following evaluation of the remaining integrals.

$$-\int_x^\infty \mathrm{d}u\, u^{-2}\,\mathrm{e}^{\mathrm{i}px} = -p\tilde{E}_2(px) \tag{34}$$

The first integral can be integrated analytically. It involves the exponential integral $\tilde{E}_2(z)$ as defined in (62):

$$E_2(-\mathrm{i}px) = -\int_x^\infty \mathrm{d}u\, u^{-2}\mathrm{e}^{\mathrm{i}pu} = -\frac{\mathrm{e}^{\mathrm{i}px}}{x} - \mathrm{i}pE_1(-\mathrm{i}px). \tag{35}$$

Inserting the boundaries and expanding the result in powers of N, our main objective is achieved. The linear term in N together with the pre-factor gives one, and the remaining terms, together with the second integral of (34) give an expression for c_0^- in closed form:

$$\lim_{N\to\infty} \langle Z^-\rangle_\varepsilon^N = \mathrm{e}^{-c_0^-(v)} \tag{36}$$

$$c_0^-(v) = \frac{\omega p\tau}{2}[\pi + E_2(pu_1) - E_2(pu_2)] + \frac{\omega\tau}{2}\left\{\int_{-\infty}^{u_1}\frac{\mathrm{d}u}{u^2}\left(1 + \frac{\tau + 2u/\tau}{\sqrt{\tau^2 + 4u(u+1)}}\right)\right.$$

$$\left. + \int_{u_2}^{\infty}\frac{\mathrm{d}u}{u^2}\left(1 - \frac{\tau + 2u/\tau}{\sqrt{\tau^2 + 4u(u+1)}}\right)\right\}\mathrm{e}^{\mathrm{i}pu}. \tag{37}$$

The further evaluation of the integral is moved to appendix B. The expression for c_0^- (74) found there reads:

$$c_0^-(v) = \frac{\kappa}{2}|r^2 - s^2| + \frac{2r^2s^2}{r^2 + s^2}\mathcal{A}_x(\sigma)\tilde{\kappa}\mathrm{e}^{-\mathrm{i}p(1-\sigma x)/2} \qquad \mathcal{A}_x(\sigma) = \frac{1}{\pi}\int_{-1}^1\frac{\mathrm{d}x\sqrt{1-x^2}}{1-\sigma x}. \tag{38}$$

The modulus $|r^2 - s^2|$ assures the invariance of $\langle Z^- \rangle^N$ with respect to the transformation $\tau \to -\tau$ mentioned in the beginning. The cusp it creates is compensated by the integral in the last term, so that $c_0^-(v)$ as a function of r and s has a continuous derivative everywhere.

Application of the functionals L_r, L_s.   Via integration by parts one can remove the differential operators, contained in the functionals. For any function $g(r)$ with $\lim_{r \to \infty} g(r) \exp(-r^2/2) = 0$, it holds

$$L_r g = 2 \int_0^\infty \mathrm{d}r \, r \mathrm{e}^{-r^2/2} g(r) - g(0). \tag{39}$$

The combination of L_r and L_s acting on a two-dimensional function $g(r, s)$, with the corresponding properties as above, can be treated accordingly. With the further assumption that $g(r, s) = g(s, r)$ one finds

$$L_r L_s g = g(0, 0) - 4 \int_0^\infty \mathrm{d}r \, r \mathrm{e}^{-r^2/2} g(r, 0) + 4\mathcal{L}g$$
$$\mathcal{L}g = \int_0^\infty \mathrm{d}r \, \mathrm{d}s \, rs \mathrm{e}^{-(r^2+s^2)/2} g(r, s). \tag{40}$$

In order to calculate the correlation function (21) we set

$$g(r, s) = \lim_{N \to \infty} \langle Z^- \rangle_{\varepsilon, v}^N = \exp(-\langle c_0^- \rangle_v) \tag{41}$$

as it is given in (38). This expression is indeed invariant with respect to exchanging the variables $r \leftrightarrow s$, so that (40) holds. $g(r, 0) = \exp(-\kappa r^2/2)$ independent of v and can be integrated giving $(1 + \kappa)^{-1}$. Subtracting the squared mean S-matrix (12) for the correlation function one gets:

$$C^-[S](\omega) = \mathcal{T} + 4\left(\mathcal{L}g - \frac{1}{1 + \kappa}\right) \tag{42}$$

$$= 4\left(\mathcal{L}g - \frac{1}{(1 + \kappa)^2}\right) \tag{43}$$

where $\mathcal{T} = 4\kappa/(1 + \kappa)^2$ is the transmission coefficient. In appendix C two alternative representations for the integral $\mathcal{L}g$ are derived: (78) and (85).

$$\mathcal{L}g = \int_0^1 \mathrm{d}\tau \int_0^\infty \mathrm{d}t \, t \exp[-t - \kappa t(\tau + \sigma^2 \mathcal{A}_x(\sigma) \langle y \mathrm{e}^{-\mathrm{i}t \frac{\kappa y}{\pi \omega}(1 - \sigma x)} \rangle_y)]. \tag{44}$$

$$\mathcal{L}g = \left(\frac{\pi \omega}{\kappa}\right)^2 \int_0^1 \mathrm{d}\tau \int_0^\infty \mathrm{d}u \, u \exp\left[-\lambda u \left(1 - \mathrm{i}\sigma \kappa' y \int_0^{uy} \frac{\mathrm{d}\rho}{\rho} \mathrm{e}^{-\mathrm{i}\rho} J_1(\sigma \rho)\right)_y\right] \tag{45}$$

with the additional parameters $\lambda = \pi \omega/\kappa'$ and $\kappa' = \kappa/(1 + \kappa)$.

5.1. General properties of the correlation function

For the Gaussian ensembles, as considered in [11], the correlation function depends only on the transmission coefficients (14). This implies its invariance under the transformation $\kappa \to \kappa^{-1}$. In contrast, the correlation function of a Poissonian ensemble does not possess this symmetry, as can be seen from the formulae (44) and (45) for $\mathcal{L}g$. Instead of $\mathcal{T}$, the relevant parameter is κ', which as a function of κ increases monotonously from zero to one. The graph of the correlation function changes accordingly, generally showing no similarity between the weak and the strong coupling limit.

The limit values of the correlation function at (a) $\omega \to 0$ and (b) $\omega \to \infty$ are independent of the distribution of the coupling matrix elements. They are calculated in the following.

(a) For $\omega \to 0$ the function $\exp\{-it\frac{\tilde{\kappa}}{\pi\omega}(1-\sigma x)\}$ in (44) becomes infinitely rapidly oscillating. Performing the integration with $\mathcal{A}_x(\sigma)$ before the averaging over the coupling matrix elements one sees that the integration of the oscillating term gives zero. Therefore, one is left with

$$\lim_{\omega \to 0} \mathcal{L}g(\omega) = \int_0^1 d\tau \int_0^\infty dt\, t e^{-t(\kappa\tau+1)} = \frac{1}{1+\kappa}. \tag{46}$$

Then (42) shows that in accordance with (14):

$$\lim_{\omega \to 0} C^-[S](\omega) = \mathcal{T}. \tag{47}$$

(b) For $\omega \to \infty$ the same exponential function as in case (a) gives one. The integral $\mathcal{A}_x(\sigma)$ then gives:

$$\mathcal{A}_x(\sigma) = \frac{1}{\pi} \int_{-1}^1 dx \frac{\sqrt{1-x^2}}{1-\sigma x} = \frac{1}{1+\tau} \tag{48}$$

and one is left with

$$\lim_{\omega \to \infty} \mathcal{L}g(\omega) = \int_0^1 d\tau \int_0^\infty dt\, t \exp\left[-t - \kappa t\left(\tau + \frac{\sigma^2}{1+\tau}\right)\right]$$

$$= \int_0^1 d\tau \int_0^\infty dt\, t e^{-t(1+\kappa)} = \frac{1}{(1+\kappa)^2}. \tag{49}$$

Then (43) shows that

$$\lim_{\omega \to \infty} C^-[S](\omega) = 0. \tag{50}$$

6. Asymptotic expansion of the correlation function

Using (45) as a starting point, $\mathcal{L}g$ is expanded in an asymptotic series in the parameter $\lambda \to \infty$ [$\lambda = \pi\omega/\kappa'$, $\kappa' = \kappa/(1+\kappa)$]. It supplies a good approximation when either ω is large or κ is small. Note that even for infinite κ, κ' stays finite (approaching one) so that the asymptotic series can still be used

$$\mathcal{L}g = \left(\frac{\pi\omega}{\kappa}\right)^2 \int_0^1 d\tau \int_0^\infty du\, u e^{-\lambda\Phi(u)}$$

$$\Phi(u) = u\left\langle 1 - i\sigma\kappa'y \int_0^{uy} \frac{d\rho}{\rho} e^{-i\rho} J_1(\sigma\rho)\right\rangle_y. \tag{51}$$

The averaging over the coupling matrix elements in the second line is changed into an averaging over the random variable y, which represents the normalized square of a coupling matrix element. The derivative of the function $\Phi(u)$ does not vanish, so a unique inverse of it always exists. The integral over u, therefore, has the only critical point $u = 0$. Using Φ as the new integration variable and performing repeated partial integrations where the exponential function is integrated and the remaining part differentiated, results in an asymptotic series for the u-integral [27]:

$$\mathcal{L}g \sim \left(\frac{\pi\omega}{\kappa}\right)^2 \int_0^1 d\tau\, e^{-\lambda\Phi(u)} \sum_{n=0}^\infty \frac{1}{\lambda^{n+1}} \left[\frac{1}{\Phi'(u)}\frac{d}{du}\right]^n \left(\frac{u}{\Phi'(u)}\right)\bigg|_{u=0}$$

$$= \left(\frac{\pi\omega}{\kappa}\right)^2 \sum_{n=0}^\infty \frac{1}{\lambda^{n+1}} s_n(0) \qquad s_n(u) = \int_0^1 d\tau \left[\frac{1}{\Phi'(u)}\frac{d}{du}\right]^n \frac{u}{\Phi'(u)}. \tag{52}$$

In principle any number of coefficients $s_n(0)$ can be calculated in a straightforward manner. But as the complexity of the corresponding expressions rapidly increases, only the first four are explicitly given. They suffice to describe the lowest-order behaviour of the correlation function $C^-[S](\omega)$ as $\omega \to \infty$.

$$s_0(0) = 0 \tag{53}$$

$$s_1(0) = 1 \tag{54}$$

$$s_2(0) = 2i\kappa'\langle y^2 \rangle \tag{55}$$

$$s_3(0) = 4\kappa'[\langle y^3 \rangle - 2\kappa'\langle y^2 \rangle^2]. \tag{56}$$

Inserting the first two coefficients explicitly, one gets

$$C^-[S](\omega) \sim \frac{4}{(1+\kappa)^2} \sum_{n=2}^{\infty} \left(\frac{\kappa'}{\pi\omega} \right)^{n-1} s_n(0). \tag{57}$$

7. Example: constant coupling matrix elements

In this section one example of a Poissonian scattering ensemble is considered. The elements of its coupling matrix (7) are all fixed to $\sqrt{\eta/N}$. Only recently have systems with constant coupling matrix elements been investigated theoretically [13, 28]. A physical system with this property is the rectangular microwave billiard with an antenna placed in its centre, where all eigenfunctions have either zero or maximal absolute values [20, 29].

Before presenting the results, some concepts will be shortly introduced in the following paragraphs, needed for the discussion below. A detailed description of them can be found in [20] and will also be published in a forthcoming paper.

Self- and pole-correlations. Starting from (1) and using the eigenbasis of H, one can decompose the S-matrix into a sum of resonance terms. Then, the product of two S-matrices (and therefore the correlation function itself) can be split into two parts: a single sum of products of resonance terms with equal indices and a double sum of products of resonance terms with distinct indices. The average of the first part, will be called 'self-correlations' whereas the average of the second 'pole-correlations'. Only the pole-correlation term is sensitive to possible correlations between different resonances. Therefore, one should try to separate it from the self-correlation term.

Rescaled Breit–Wigner approximation. For sufficiently weak coupling, one can apply first-order perturbation theory for V in (2). Then, the positions and widths of the resonances are directly given by the diagonal matrix elements of H_0 and $VV^\dagger$. In this approximation the ensemble averaging can be easily done. The self-correlation term depends only on the distribution of the coupling matrix elements, whereas the pole-correlation term additionally depends on the correlations in the eigenvalues of H_0; consequently, the latter vanishes for Poissonian ensembles.

But, even for moderate coupling strengths ($\kappa \approx 0.1$), a phenomenological rescaling is necessary if a good approximation of the correlation function is required. The self- and pole-correlation terms, obtained in this way, can be identified with analogous results from higher-order perturbative calculations [30]. For the Poissonian ensemble (only self-correlations), the rescaled Breit–Wigner approximation of the correlation function is:

$$C^-[S](\omega) = \mathcal{T} \left\langle \frac{y^2}{y - \frac{2i\pi\omega}{\mathcal{T}}} \right\rangle_y. \tag{58}$$

Note that here the complex conjugate of the usual definition is used (cf the discussion below (20)). The random variable y represents the normalized square of a coupling matrix element. In the case of constant coupling matrix elements, $y = 1$, and the correlation function becomes a Lorentzian with the width:

$$\frac{\Gamma_C}{d} = \frac{\mathcal{T}}{2\pi} \tag{59}$$

which is in perfect agreement with the theory of Ericson fluctuations [31, 32], where Γ_C is called 'correlation length'.

Correlation hole. Here, the 'correlation hole' serves as a means to detect pole-correlations in the correlation function (in this sense it is used in connection with the Fourier transform of correlation functions [8]). It is defined as a minimum in the real part of the correlation function. Note however, that the appearance of a correlation hole is a sufficient (but not a necessary) condition for the existence of pole-correlations. The reason is, that the self-correlation term alone is a strictly decreasing function of $0 < \omega < \infty$ for any coupling strength [20].

In the following the correlation function is calculated and discussed for three different coupling strengths. Figure 2 shows $C^-[S](\omega)$ for the case of small but still so strong coupling, so that the beginning of the break-down of the rescaled Breit–Wigner approximation can be seen. Figure 3 shows the correlation function in the strong coupling case, where the correlation hole has fully developed. Finally, figure 4 shows the case where the highest-order coefficient $s_3(0)$, (56), of the asymptotic expansion for the real part of the correlation function just vanishes.
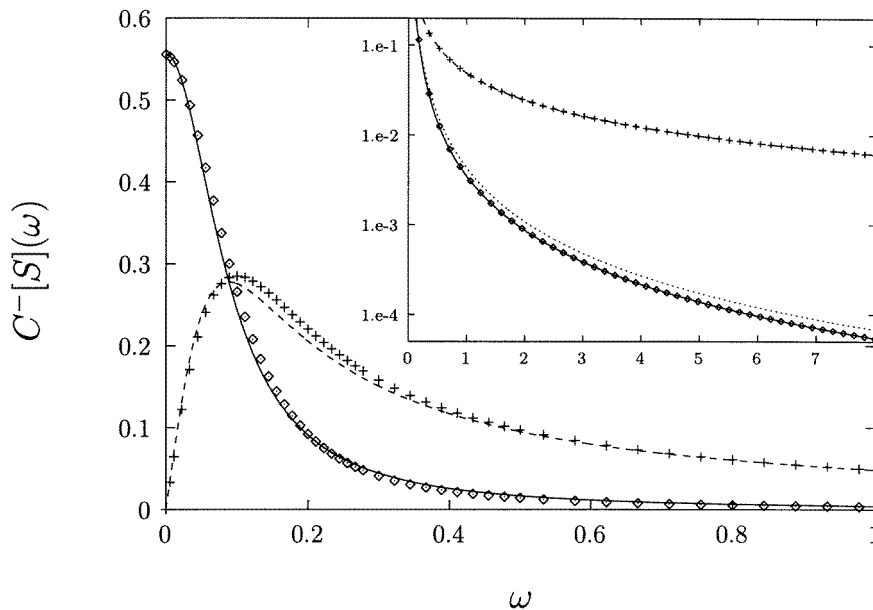

Figure 2. $C^-[S]$ in the case of constant coupling matrix elements; $\kappa = 0.2$. Comparison with the asymptotic expansion (57) and the rescaled Breit–Wigner approximation (58). Real (diamonds) and imaginary (crosses) parts of the exact correlation function using (44). In the main plot, the real (full curve) and imaginary (broken curve) parts of the rescaled Breit–Wigner approximation are shown. In the inset, the leading terms ($n = 2, 3$) of the real (full curve) and imaginary (broken curve) parts of the asymptotic expansion; the dotted curve gives the real part of the rescaled Breit–Wigner approximation.
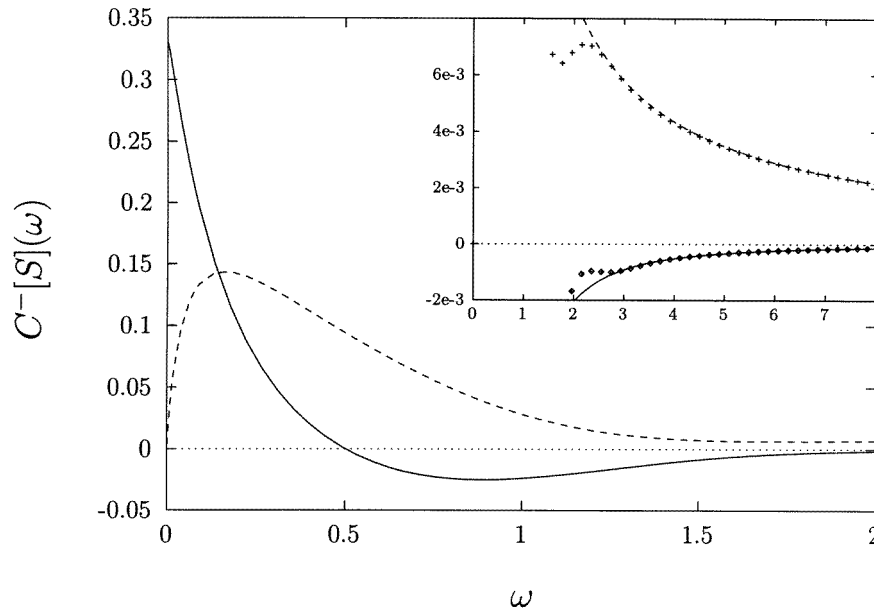
Figure 3. $C^-[S]$ in the case of constant coupling matrix elements; $\kappa = 10$. Full curve in the main plot and diamonds in the inset: real part of the exact correlation function using (44); broken curve in the main plot and crosses in the inset: corresponding imaginary part. Full curve in the inset: real part ($n = 3$) of the asymptotic expansion (57); broken curve in the inset: corresponding imaginary part ($n = 2$).

In figure 2, $C^-[S](\omega)$ is shown for $\kappa = 0.2$. The coupling strength chosen is just strong enough, so that the exact correlation function starts to differ significantly from the rescaled Breit–Wigner approximation (58), which simply gives the real and imaginary part of a Lorentzian with the width Γ_C/d as given in (59). The deviation from the Lorentzian signals the beginning of a re-organization process. The resonance poles start to interfere with each other [24], 'trying' to stay orthogonal to each other and at the same time try to align with the channel vector (finally, this will result in a second-order phase transition as discussed in [25]).

For small ω the difference between the exact and the approximate correlation function is strongest in the imaginary part of $C^-[S](\omega)$ as can be seen in the main plot of figure 2. On the other hand, for large ω, the difference is strongest in the real part. As demonstrated in the inset of figure 2, for Im $C^-[S](\omega)$ the rescaled Breit-Wigner curve perfectly meets the exact one, whereas for Re $C^-[S](\omega)$ both curves already differ in the lowest-order expansion coefficient.

Note, that the asymptotic expansion in lowest order is already quite accurate, starting from $\omega \approx 1$.

In figure 3 the case of strong coupling $\kappa = 10$ is shown. All (except one) resonance poles are again lying close to the real axis and would, with increasing coupling strength, further approach it. In principle, again one has well separated Lorentzian shaped resonances, so that the rescaled Breit–Wigner approximation could be applied. However, this is not feasible because the closed system to which the effective Hamiltonian H (projected on a subspace perpendicular to the channel vector) converges is different from H_0, and generally has very different statistical properties. Namely, it exhibits strong pole-correlations.

For large ω there is still a good agreement between the exact and the asymptotic result, although starting from a somewhat larger value $\omega > 3$ as in the small coupling case. The
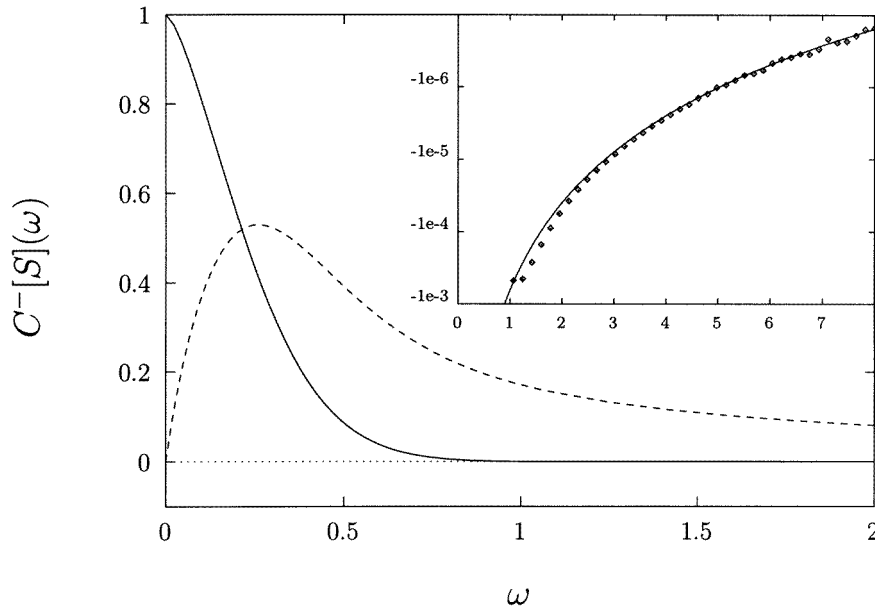
Figure 4. $C^-[S]$ in the case of constant coupling matrix elements; $\kappa = 1$. Full curve in the main plot and diamonds in the inset: real part of the exact correlation function using (44); broken curve in the main plot: corresponding imaginary part. Full curve in the inset: real part ($n = 5$) of the asymptotic expansion (57).

appearance of a correlation hole does not affect the validity of the asymptotic expansion; on the contrary, the correlation hole can be deduced from the fact that the lowest-order expansion coefficient $s_3(0)$ is negative, because the real part of the correlation function then has to approach the ω-axis from below, from which the existence of a minimum follows. As discussed above this correlation hole proves the existence of pole-correlations.

The correlation hole depicted in figure 3 is well pronounced, and in so far it fits to the observations made in [6, 13–15]. There, it was found that the correlations between the S-matrix poles increase with the coupling strength until settlement in the strong coupling limit.

From (56) it is found, that a correlation hole is present as soon as $\kappa > 1$ (for constant coupling matrix elements) and as $\kappa > 5$ (for Gaussian distributed coupling matrix elements). In the latter case κ is five times larger, suggesting a less pronounced correlation hole in the strong coupling limit.

The relation (56) furthermore shows, that distributions of coupling matrix elements exist, for which the expansion coefficient $s_3(0)$ never becomes negative. This would be an even stronger objection against the formation of a correlation hole. A limiting case, where no correlations are produced at all is considered in [13, 20].

In figure 4 the coupling strength is chosen, such that the lowest-order coefficient $s_3(0)$ of the asymptotic expansion of the real part of the correlation function just vanishes. For the case of constant coupling matrix elements this point is identical to the critical point of maximal transmission. Then, the real part of the correlation function is of order ω^{-4}, governed by the coefficient $s_5(0)$:

$$s_5(0) = 4\kappa'(-4 + 55\kappa' - 144\kappa'^2 + 96\kappa'^3) \tag{60}$$

which gives

$$s_5(0) = -1 \tag{61}$$

at the critical point $\kappa = 1$. This surprising result proves the existence of a correlation hole already at the critical point (and due to continuity, also in a small neighbourhood around it). As one can see in the inset of figure 4, the behaviour of $\text{Re}\, C^-[S](\omega)$ at large ω is indeed correctly described by the asymptotic expansion, using (61). However, the correlation hole is very small.

8. Summary

In this paper a non-perturbative method for dealing with Poissonian scattering ensembles has been developed. It allows one to average analytically over products of *S*-matrices. This is by itself an important achievement, as until now only a few scattering ensembles could be treated analytically; amongst them the scattering ensembles based on Gaussian random Hamiltonians. In contrast to the Gaussian ensembles, the Poissonian ensembles should be used for the statistical description of generically integrable systems.

The comparison of the correlation function for the two types of ensembles, clarifies the role played by the level-correlations for the correlation properties of the scattering cross section. For instance, it turns out that the level-correlations are irresponsible for the emergence of Ericson fluctuations which are solely produced by the self-correlation term. This is discussed more extensively in [20].

One very interesting feature of the correlation function is a 'correlation hole' which may occur, when the coupling to the decay channels is sufficiently strong. It proves correlations between different resonances (pole-correlations). In this way, pole-correlations were found in Poissonian scattering systems, which had been produced by the coupling to the decay channels. It was further shown that they increase with the coupling strength and finally saturate. These findings confirm earlier numerical investigations of the correlations between *S*-matrix poles [14, 15]. In the strong coupling limit, one arrives at a new closed system (of dimension $N - 1$) with totally different correlation properties. This behaviour should be contrasted to the GOE case, where the correlation function is invariant under the transformation $\kappa \to \kappa^{-1}$. In the weak as well as in the strong coupling limit, the correlation function reflects the same level-correlations of the GOE-type, whereas, with increasing transmission, the correlations get washed out.

In the case of constant coupling matrix elements, it could be demonstrated, that pole correlations appear even before the critical point is reached. Even though the correlation hole was found to be extremely small, this is an important result in view of more principle questions. First of all one could have thought of finding some signature of the critical point in the correlation function, which is apparently not the case. Second, it is doubtful whether one can find physical systems with an external parameter to drive them across the critical point. Then, one could have argued, that correlations can be induced only when the coupling is passing its physical limits (the critical point), so that the statement 'the coupling produces correlations' was irrelevant.

In order to test the results presented in this paper, on a real physical system, one could follow two lines: (a) one could perform an appropriate experiment, similar to [2], with a superconducting rectangular microwave billiard, recording the reflection spectrum and analysing the correlations within; (b) alternatively one could conduct a numerical analysis. Then a two-dimensional rectangular microwave billiard coupled to a coplanar waveguide would probably be more appropriate. Studies on such a system (although at first with different interests) have recently started [33].

Appendix A. A special kind of the exponential integral

For the integration of the first integrand of (34) an anti-derivative of $x^{-2}e^{ix}$ is needed, which should be analytic at infinity. This is accomplished by the following definition

$$\tilde{E}_2(z) = \int_z^\infty du \frac{e^{iu}}{u^2} = \frac{e^{iz}}{z} + iE_1(-iz) \qquad |\arg(-iz)| < \pi. \tag{62}$$

The integration path may not cross the negative real axis including the origin. E_1 is the exponential integral according to [26]. For asymptotically small real arguments x, one gets

$$\tilde{E}_2(x) = \frac{1}{x} + i[1 - \gamma - \ln(-ix)] + O(x)$$

$$= \frac{1}{x} + i\left[1 - \gamma - \ln|x| + \frac{i\pi}{2}\operatorname{sgn}(x)\right] + O(x). \tag{63}$$

γ is Euler's constant.

Appendix B. Simplification of $c_0^-(v)$

Both integrals in (37) will be transformed in such a way that each of the radicands will become $x^2 - 1$. So in the first integral the transformation reads: $u = (\sigma x + 1)/2$ and in the second integral: $u = (\sigma x - 1)/2$. Due to the transformations both integration intervals are mapped onto the interval: $(-1, 1)$. A further transformation $x \to -x$ in the second integral allows one to combine both integrals into a single one:

$$c_0^-(v) = \frac{\omega p \tau}{2}[\pi + \tilde{E}_2(pu_1) - \tilde{E}_2(pu_2)] + F(p, \tau)$$

$$F(p, \tau) = -\omega\sigma e^{-ip/2} \int_1^\infty \frac{dx}{\sqrt{1-x^2}} \left\{ \frac{\tau\sqrt{x^2-1} - x - \sigma}{(\sigma x + 1)^2} e^{-i\beta x} + \frac{\tau\sqrt{x^2-1} - x + \sigma}{(\sigma x - 1)^2} e^{i\beta x} \right\}$$

$$\tag{64}$$

where $\beta = p\sigma/2$. Discarding the exponentials, each of the two terms can be integrated as a indefinite integral, using standard techniques [34]. Note that both terms in the curly brackets differ only in the sign in front of the parameter σ. Considering both cases as anti-derivatives one gets

$$\int^x ds \frac{\tau\sqrt{s^2-1} - (s \pm \sigma)}{\sqrt{s^2-1}(\sigma s \pm 1)^2} = \frac{2}{1+\tau}\left(x + \sqrt{x^2-1} \pm q\right)^{-1} \tag{65}$$

where $q = s/r$. With this result, one can integrate partially in F, and arrive at

$$F(p, \tau) = \frac{2\omega\sigma}{1+\tau} e^{-ip/2} \left\{ \frac{e^{-i\beta}}{1+q} + \frac{e^{i\beta}}{1-q} + i\beta A \right\} \tag{66}$$

$$A = \int_1^\infty dx \left(\frac{e^{i\beta x}}{x + \sqrt{x^2-1} - q} - \frac{e^{-i\beta x}}{x + \sqrt{x^2-1} + q} \right). \tag{67}$$

After the substitution $u = -x - \sqrt{x^2 - 1}$ in the first integrand and $u = x + \sqrt{x^2 - 1}$ in the second, one can recombine the integrals into one:

$$A = \left[\int_{-\infty}^{-1} + \int_1^\infty \right] du \, \frac{u - u^{-1}}{2u(u-q)} e^{i\beta(u+u^{-1})/2}. \tag{68}$$

The integration intervals form a part of a closed path in the complex plane, consisting of the interval $(-\infty, -1)$, the upper half unit circle from -1 to 1, the interval $(1, \infty)$, and the upper half circle from ∞ to $-\infty$. The integrand is analytic in the region enclosed by this path, and the integral over the semicircle with 'infinite' radius gives zero. Therefore one gets

$$A = -\int_0^\pi d\phi \, \frac{\sin\phi \, e^{i\beta\cos\phi}}{e^{i\phi} - q} = -\int_{-1}^1 \frac{dx \, e^{i\beta x}}{x - q + i\sqrt{1 - x^2}} = \frac{\sigma}{2q}(A_1 + iA_2) \tag{69}$$

$$A_1 = \int_{-1}^1 dx \, e^{i\beta x} \frac{q - x}{1 - \sigma x} = \frac{2\sin\beta}{\sigma\beta} + \frac{\tau}{\sigma^2} e^{ip/2} \left\{ E_1 \left[\frac{ip}{2}(1+\sigma) \right] - E_1 \left[\frac{ip}{2}(1-\sigma) \right] \right\} \tag{70}$$

$$A_2 = \int_{-1}^1 dx \, e^{i\beta x} \frac{\sqrt{1 - x^2}}{1 - \sigma x}. \tag{71}$$

Inserting the results for A_1 (70), for $F(p, \tau)$ (66) into the starting expression for $c_0^-(v)$ (64) one gets

$$c_0^- = \pi\omega p\tau/2 + \omega e^{-ip/2} \left[\frac{\tau e^{i\beta}}{1 - \sigma} - \frac{\tau e^{-i\beta}}{1 + \sigma} - \frac{2\sigma}{1 + \tau} \left(\frac{e^{-i\beta}}{1+q} + \frac{e^{i\beta}}{1-q} + \frac{i\sin\beta}{q} \right) \right]$$
$$- \left[i\omega p\tau/2 - \frac{2\omega\sigma}{1 + \tau} \frac{i\beta\tau}{2q\sigma} \right] \left\{ E_1 \left[\frac{ip}{2}(1+\sigma) \right] - E_1 \left[\frac{ip}{2}(1-\sigma) \right] \right\}$$
$$- \frac{2\omega\sigma}{1 + \tau} \frac{\beta\sigma}{2q} e^{-ip/2} A_2. \tag{72}$$

Replacing the parameters p, τ, σ, and q by their definitions (25) and (26) as functions depending on r and s, it turns out, that the expressions in both square brackets vanish.

$$c_0^- = \frac{\tilde\kappa}{2}|r^2 - s^2| + \tilde\kappa \frac{2r^2s^2}{r^2 + s^2} \mathcal{A}_x(\sigma) e^{-ip(1-\sigma x)/2} \tag{73}$$

$$\mathcal{A}_x(\sigma) = \frac{1}{\pi} \int_{-1}^1 dx \, \frac{\sqrt{1 - x^2}}{1 - \sigma x}. \tag{74}$$

Appendix C. Simplified expressions for the integral $\mathcal{L}g$

First representation.   In order to calculate the correlation function $C^-[S](\omega)$ one needs the integral $\mathcal{L}g$ as it is given due to the relations (38), (40) and (41). Using the abbreviation $R^2 = r^2 + s^2$, and the parameters σ, τ, and κ as defined in (25) and (26) one finds

$$\mathcal{L}g = \int\!\!\int_0^\infty dr \, ds \, rs \exp\left[-\frac{R^2}{2}(1 + \kappa|\tau|) + \sigma^2 \mathcal{A}_x(\sigma) \langle \tilde\kappa e^{-ip(1-\sigma x)/2} \rangle_v \right]. \tag{75}$$

This expression can be simplified further by introducing spherical coordinates: $r = R\cos\phi$, $s = R\sin\phi$. Then it holds:

$$r^2 + s^2 = R^2 \qquad \sigma = 2\cos\phi\sin\phi = \sin 2\phi \qquad \tau = \cos^2\phi - \sin^2\phi = \cos 2\phi. \tag{76}$$

Because $rs = R^2\sigma/2$ and $p = R^2\tilde\kappa/(\pi\omega)$ the transition to the new coordinates is easily accomplished. The integral operator $\mathcal{L}$ alone, now reads:

$$\mathcal{L} = \tfrac{1}{2} \int_0^\infty dR \, R^3 e^{-R^2/2} \int_0^{\pi/2} d\phi \, \sin 2\phi = \tfrac{1}{2} \int_0^\infty dt \, t e^{-t} \int_0^\pi d\theta \, \sin\theta \tag{77}$$

with $\theta = 2\phi$ and $t = R^2/2$. Because $g(r, s)$ is symmetric with respect to the transformation $\theta \to \pi - \theta$, one can restrict the integration to the half of the original integration interval, namely to $(0, \pi/2)$. There a further substitution $\theta = \cos\phi$ is applied. Finally $\tilde\kappa$ is replaced by κy due to the relation (25). y is a random variable for the normalized square of a coupling matrix element, which coincides in the weak coupling limit with the normalized width of the resonances. We then get the first of the two representations of $\mathcal{L}g$ which are used in sections 5 and 8:

$$\mathcal{L}g = \int_0^1 d\tau \int_0^\infty dt\, t \exp[-t - \kappa t(\tau + \sigma^2 \mathcal{A}_x(\sigma)\langle y e^{-it\frac{\kappa y}{\pi\omega}(1-\sigma x)}\rangle_y)]. \tag{78}$$

Second representation. A second representation of $\mathcal{L}g$ can be found by rewriting the integral

$$\mathcal{A}_x(\sigma)e^{-ip(1-\sigma x)/2} = \frac{1}{\pi}\int_{-1}^1 dx\, \sqrt{1-x^2}\frac{e^{-ip(1-\sigma x)/2}}{1-\sigma x}. \tag{79}$$

The latter part of the integrand can be expressed by the following indefinite integral:

$$f(p) = \frac{e^{-ip(1-\sigma x)/2}}{1-\sigma x} = \frac{i}{2}\int_p^{-i\infty} d\rho\, e^{-i\rho(1-\sigma x)/2}. \tag{80}$$

Inserting $f(p)$ into (79) and exchanging the order of integrations leads to

$$\mathcal{A}_x(\sigma)e^{-ip(1-\sigma x)/2} = \frac{i}{2}\int_p^{-i\infty} d\rho\, e^{-i\rho/2}\frac{1}{\pi}\int_{-1}^1 dx\, \sqrt{1-x^2}e^{-i\sigma\rho x/2}. \tag{81}$$

The x-integral is a representation of the Bessel function of the first kind, as can be realized after the substitution $x = \cos\phi$ and subsequent partial integration [26]

$$\frac{1}{\pi}\int_{-1}^1 dx\, \sqrt{1-x^2}e^{-i\sigma\rho x/2} = \frac{2i}{\pi\sigma\rho}\int_{-\pi/2}^{\pi/2} d\phi\, \sin\phi e^{-i\sigma\rho\sin\phi/2} = \frac{2}{\sigma\rho}J_1(\sigma\rho/2). \tag{82}$$

Therefore, we finally have

$$\mathcal{A}_x(\sigma)e^{-ip(1-\sigma x)/2} = \frac{i}{\sigma}\int_{p/2}^{-i\infty} \frac{d\rho}{\rho}e^{-i\rho}J_1(\sigma\rho). \tag{83}$$

As for $p = 0$, we have $\sigma^2\mathcal{A}_x(\sigma) = 1 - \tau$ (cf (48)) one can split the integration path to get

$$i\sigma\int_{p/2}^{-i\infty} \frac{d\rho}{\rho}e^{-i\rho}J_1(\sigma\rho) = i\sigma\left[\int_0^{-i\infty} - \int_0^{p/2}\right]\frac{d\rho}{\rho}e^{-i\rho}J_1(\sigma\rho)$$

$$= (1-\tau) - i\sigma\int_0^{p/2} \frac{d\rho}{\rho}e^{-i\rho}J_1(\sigma\rho). \tag{84}$$

Inserting this expression in (78) and using the substitution $u = \kappa t/(\pi\omega)$ finally gives

$$\mathcal{L}g = \left(\frac{\pi\omega}{\kappa}\right)^2 \int_0^1 d\tau \int_0^\infty du\, u \exp\left[-\lambda u\left\langle 1 - i\sigma\kappa'y\int_0^{uy} \frac{d\rho}{\rho}e^{-i\rho}J_1(\sigma\rho)\right\rangle_y\right]. \tag{85}$$

Here the additional parameters $\lambda = \pi\omega/\kappa'$ and $\kappa' = \kappa/(1+\kappa)$ are introduced.

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